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Topological structure of solution sets to asymptotic boundary value problems ☆

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ABSTRACT

Topological structure is investigated for second-order vector asymptotic boundary value problems. Because of indicated obstructions, the R_δ-structure is firstly studied for problems on compact intervals and then, by means of the inverse limit method, on non-compact intervals. The information about the structure is furthermore employed, by virtue of a fixed-point index technique in Fréchet spaces developed by ourselves earlier, for obtaining an existence result for nonlinear asymptotic problems. Some illustrating examples are supplied.

1. Introduction

Cauchy (initial value) problems for ordinary differential equations are, according to the result of Orlicz, generically solvable in a unique way. For the exceptional cases (non-uniqueness), Kneser firstly proved that the sets of their solutions are at every fixed time continua and then Hukuhara showed that the solution set (on a compact interval) itself is a continuum. Aronszajn improved this result in the sense that the solution sets are compact and acyclic, but in fact he specified these continua to be

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R_δ-sets. The analogous result was obtained for upper-Carathéodory differential inclusions by De Blasi and Myjak in [13]. For more details, historical remarks and related references, see [4, III.12.2].

Topological structure of solution sets to Cauchy problems on non-compact and, in particular, infinite intervals was studied by various techniques, e.g., in [1,4,9,12,16,21,28,29].

For boundary value problems, the situation is much more delicate and the related results are still very rare; see, e.g., [4, Chapter III.3], [8,14,24]. So far, topological structure of solution sets was investigated exclusively (as far as we know, with only one exception [19]) to boundary value problems on compact intervals. Moreover, because of the counter-examples in [2,16], [4, Example II.2.12], demonstrating the impossibility of asymptotic analogies to the situation on compact intervals, the main theorem in [19] might be empty. These troubles are due to an “unpleasant” related topology of non-normable Fréchet spaces. For instance, a contractivity of a given operator with respect to a metric need not follow from a contractivity with respect to each seminorm. Moreover, bounded subsets of non-normable Fréchet spaces have always empty interiors, etc.

Despite these difficulties, there is a chance to obtain some results for at least particular asymptotic problems like Kneser-type (Thomas–Fermi) problems. The key tool is for us the inverse limit method, sometimes also called the projective limit (see, e.g., [2–4,9,15,18,25]). We elaborated this technique for the needs of multivalued analysis in [1,2], [4, Chapter II.2]. We believe that this approach can bring further impulses in the field reflected in the title.

2. Preliminaries

At first, we recall some geometric notions of subsets of metric spaces, in particular, of retracts. For more details, see, e.g., [4,10,17].

For a subset $A \subset X$ of a metric space $X = (X, d)$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A\colon d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$.

We say that a metric space X is an *absolute retract* (AR-space) if, for each metric space Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. Let us note that X is an AR-space if and only if it is a retract of some normed space. Moreover, if X is a retract of a convex set in a Fréchet space, then it is an AR-space. So, in particular, the spaces $C(J, \mathbb{R}^n)$, $C^1(J, \mathbb{R}^n)$, $AC^1_{loc}(J, \mathbb{R}^n)$ are AR-spaces as well as their convex subsets, where $J \subset \mathbb{R}$ is an arbitrary interval. The foregoing symbols denote, as usually, the spaces of functions $f : J \to \mathbb{R}^n$ which are continuous, smooth and those with locally absolutely continuous first derivatives, respectively, endowed with the respective topologies.

We say that a nonempty subset A of a metric space X is *contractible* if there exist a point $x_0 \in A$ and a homotopy $h : A \times [0,1] \to A$ such that $h(x,0) = x$ and $h(x,1) = x_0$, for every $x \in A$. A nonempty set $A \subset X$ is called an *R_δ-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact, AR-spaces (or, despite of the hierarchy (1) below, compact, contractible sets) such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Note that any R_δ-set is nonempty, compact and connected. The following hierarchy holds for nonempty subsets of a metric space:

$$\text{compact} + \text{convex} \subset \text{compact } AR\text{-space} \subset \text{compact} + \text{contractible} \subset R_\delta\text{-set}, \tag{1}$$

and all the above inclusions are proper.

We also employ the following definitions and statements from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that F is a multivalued mapping from X to Y

(written $F : X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X. Every upper semicontinuous map with closed values has a closed graph.

The reverse relation between upper semicontinuous mappings and those with closed graphs is expressed in the following proposition.

Proposition 2.1. *(Cf., e.g., [4,17].) Let X, Y be metric spaces and $F : X \multimap Y$ be a multivalued mapping with the closed graph such that $F(X) \subset K$, where K is a compact set. Then F is u.s.c.*

A multivalued mapping $F : X \multimap X$ with bounded values is called *Lipschitzian* if there exists a constant $L > 0$ such that

$$d_H\big(F(x), F(y)\big) \leqslant L\, d(x, y),$$

for every $x, y \in X$, where

$$d_H(A, B) := \inf\{r > 0 \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

stands for the Hausdorff distance; for its properties, see, e.g., [4,17].

We say that a multivalued mapping $F : X \multimap X$ with bounded values is a *contraction* if it is Lipschitzian with a Lipschitz constant $L \in [0, 1)$.

Let Y be a separable metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space*, i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $F : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

We say that mapping $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$, is an *upper-Carathéodory mapping* if the map $F(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable on every compact subinterval of J, for all $x \in \mathbb{R}^m$, the map $F(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in J$, and the set $F(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

We will employ the following selection statement.

Proposition 2.2. *(Cf., e.g., [6].) Let $F : [a, b] \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping satisfying $|y| \leqslant r(t)(1 + |x|)$, for every $(t, x) \in [a, b] \times \mathbb{R}^m$, and every $y \in F(t, x)$, where $r : [a, b] \to [0, \infty)$ is an integrable function. Then the composition $F(t, q(t))$ admits, for every $q \in C([a, b], \mathbb{R}^m)$, a single-valued measurable selection.*

If $X \cap Y \neq \emptyset$ and $F : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed-point* of F if $x \in F(x)$. The set of all fixed-points of F will be denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}.$$

It will be also convenient to recall the following results.

Proposition 2.3. *(Cf. [26].) Let X be a closed, convex subset of a Banach space E and let $\phi : X \multimap X$ be a contraction with compact, convex values. Then $\mathrm{Fix}(\phi)$ is a nonempty, compact AR-space.*

Lemma 2.1. *(Cf. [7, Theorem 0.3.4].) Let $[a, b] \subset \mathbb{R}$ be a compact interval. Assume that the sequence of absolutely continuous functions $x_k : [a, b] \to \mathbb{R}^n$ satisfies the following conditions:*

(i) *the set* $\{x_k(t) \mid k \in \mathbb{N}\}$ *is bounded, for every* $t \in [a,b]$,
(ii) *there exists a function* $\alpha : [a,b] \to \mathbb{R}$, *integrable in the sense of Lebesgue, such that*

$$\left|\dot{x}_k(t)\right| \leqslant \alpha(t), \quad \textit{for a.a. } t \in [a,b] \textit{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence of $\{x_k\}$ (*for the sake of simplicity denoted in the same way as the sequence*) *converging to an absolutely continuous function* $x : [a,b] \to \mathbb{R}^n$ *in the following way*:

1. $\{x_k\}$ *converges uniformly to* x,
2. $\{\dot{x}_k\}$ *converges weakly in* $L^1([a,b], \mathbb{R}^n)$ *to* $\dot{x}$.

The following lemma is a slight modification of the well-known result.

Lemma 2.2. *(Cf. [30, p. 88].) Let* $[a,b] \subset \mathbb{R}$ *be a compact interval,* E_1, E_2 *be Euclidean spaces and* $F : [a,b] \times E_1 \multimap E_2$ *be an upper-Carathéodory mapping.*

Assume in addition that, for every nonempty, bounded set $\mathcal{B} \subset E_1$, *there exists* $\nu = \nu(\mathcal{B}) \in L^1([a,b], [0,\infty))$ *such that*

$$\left|F(t,x)\right| \leqslant \nu(t),$$

for a.a. $t \in [a,b]$ *and every* $x \in \mathcal{B}$.

Let us define the Nemytskiĭ operator $N_F : C([a,b], E_1) \multimap L^1([a,b], E_2)$ *in the following way*:

$$N_F(x) := \left\{ f \in L^1\big([a,b], E_2\big) \;\middle|\; f(t) \in F\big(t, x(t)\big), \textit{ a.e. on } [a,b] \right\},$$

for every $x \in C([a,b], E_1)$. *Then, if sequences* $\{x_i\} \subset C([a,b], E_1)$ *and* $\{f_i\} \subset L^1([a,b], E_2)$, $f_i \in N_F(x_i)$, $i \in \mathbb{N}$, *are such that* $x_i \to x$ *in* $C([a,b], E_1)$ *and* $f_i \to f$ *weakly in* $L^1([a,b], E_2)$, *then* $f \in N_F(x)$.

3. Topological structure on compact intervals

Before investigating the asymptotic problems, it will be useful to study the topological structure of related solution sets on compact intervals.

At first, let us consider the problems for fully linearized systems

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C(t), \quad & \text{for a.a. } t \in [0,m], \\ x \in S_m, \quad & \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in C(t), \quad & \text{for a.a. } t \in [0,m], \\ (x, \dot{x}) \in S'_m, \quad & \end{aligned}\right\} \tag{3}$$

where

(i) $A, B : [0,m] \to \mathbb{R}^{n \times n}$ are integrable matrix functions such that $|A(t)| \leqslant a(t)$, $|B(t)| \leqslant b(t)$, for a.a. $t \in [0,m]$ and suitable nonnegative functions $a, b \in L^1([0,m], \mathbb{R})$,
(ii) S_m is a closed, convex subset of $AC^1([0,m], \mathbb{R}^n)$ (S'_m is a closed, convex subset of $AC^1([0,m], \mathbb{R}^n) \times AC([0,m], \mathbb{R}^n)$),
(iii) $C : [0,m] \multimap \mathbb{R}^n$ is an integrable mapping with convex closed values such that $|C(t)| \leqslant c(t)$, for a.a. $t \in [0,m]$ and a suitable nonnegative function $c \in L^1([0,m], \mathbb{R})$,
(iv) there exist $t_0 \in [0,m]$ and constants M_0, M_1 such that $|x(t_0)| \leqslant M_0$ and $|\dot{x}(t_0)| \leqslant M_1$, for all solutions of problem (2) (all solutions of problem (3)).

Lemma 3.1. *Under the above assumptions* (i)–(iv)*, the solution set of problem* (2) (*the set of solutions and their first derivatives of problem* (3)) *is convex and compact.*

Proof. Let us prove that the set of solutions and their first derivatives of the b.v.p. (3) is convex and compact. By the similar reasoning, it is possible to obtain that the solution set of problem (2) is convex and compact as well.

Let us denote by $P(t, x(t), \dot{x}(t)) := C(t) - A(t)\dot{x}(t) - B(t)x(t)$. If x_1, x_2 are solutions of problem (3), then it follows from the integral representation of a solution and its derivative that, for a.a. $t \in [0, m]$, we have

$$x_1(t) \in x_1(t_0) + \dot{x}_1(t_0) \cdot (t - t_0) + \int_{t_0}^{t} (t - s) \cdot P\big(s, x_1(s), \dot{x}_1(s)\big)\, ds,$$

$$x_2(t) \in x_2(t_0) + \dot{x}_2(t_0) \cdot (t - t_0) + \int_{t_0}^{t} (t - s) \cdot P\big(s, x_2(s), \dot{x}_2(s)\big)\, ds$$

and

$$\dot{x}_1(t) \in \dot{x}_1(t_0) + \int_{t_0}^{t} P\big(s, x_1(s), \dot{x}_1(s)\big)\, ds,$$

$$\dot{x}_2(t) \in \dot{x}_2(t_0) + \int_{t_0}^{t} P\big(s, x_2(s), \dot{x}_2(s)\big)\, ds.$$

Let $\theta \in [0, 1]$ be arbitrary. Then

$$\begin{aligned}
&\theta x_1(t) + (1-\theta)x_2(t) \\
&\quad \in \theta \cdot x_1(t_0) + (1-\theta) \cdot x_2(t_0) + \big[\theta \cdot \dot{x}_1(t_0) + (1-\theta) \cdot \dot{x}_2(t_0)\big] \cdot (t - t_0) \\
&\qquad + \int_{t_0}^{t} (t-s) \cdot \theta \cdot P\big(s, x_1(s), \dot{x}_1(s)\big)\, ds + \int_{t_0}^{t} (t-s) \cdot (1-\theta) \cdot P\big(s, x_2(s), \dot{x}_2(s)\big)\, ds \\
&\quad = \theta \cdot x_1(t_0) + (1-\theta) \cdot x_2(t_0) + \big[\theta \cdot \dot{x}_1(t_0) + (1-\theta) \cdot \dot{x}_2(t_0)\big] \cdot (t - t_0) \\
&\qquad + \int_{t_0}^{t} (t-s) \cdot P\big(s, \theta x_1(s) + (1-\theta)x_2(s), \theta \dot{x}_1(s) + (1-\theta)\dot{x}_2(s)\big)\, ds.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\theta \dot{x}_1(t) + (1-\theta)\dot{x}_2(t) &\in \theta \dot{x}_1(t_0) + (1-\theta)\dot{x}_2(t_0) \\
&\quad + \int_{t_0}^{t} P\big(s, \theta x_1(s) + (1-\theta)x_2(s), \theta \dot{x}_1(s) + (1-\theta)\dot{x}_2(s)\big)\, ds.
\end{aligned}$$

Finally, because of convexity of S'_m, we obtain that

$$\big(\theta x_1 + (1-\theta)x_2, \theta\dot{x}_1 + (1-\theta)\dot{x}_2\big) \in S'_m$$

and, therefore, the set of solutions of (3) and their derivatives is convex.

Let us also prove that the set of solutions of (3) and their derivatives is relatively compact. It follows from the well-known Arzelà–Ascoli lemma that the set of solutions is relatively compact in $C^1([0,m],\mathbb{R}^n)$ if and only if it is bounded and all solutions and their first derivatives are equi-continuous.

At first, let us show that the set of solutions of (3) is bounded in $C^1([0,m],\mathbb{R}^n)$. Let x be a solution of (3) and let $t \in [0,m]$ be arbitrary. Then

$$\begin{aligned}
\big|x(t)\big| + \big|\dot{x}(t)\big| &\leqslant \big|x(t_0)\big| + \big|\dot{x}(t_0)\big| \cdot |t-t_0| + \left|\int_{t_0}^{t} |t-s| \cdot \big|P\big(s,x(s),\dot{x}(s)\big)\big|\,ds\right| + \big|\dot{x}(t_0)\big| \\
&\quad + \left|\int_{t_0}^{t} \big|P\big(s,x(s),\dot{x}(s)\big)\big|\,ds\right| \\
&\leqslant M_0 + M_1 \cdot |t-t_0| + |t-t_0| \cdot \left|\int_{t_0}^{t} c(s) + a(s)\big|\dot{x}(s)\big| + b(s)\big|x(s)\big|\,ds\right| + M_1 \\
&\quad + \left|\int_{t_0}^{t} c(s) + a(s)\big|\dot{x}(s)\big| + b(s)\big|x(s)\big|\,ds\right| \\
&\leqslant M_0 + M_1 \cdot [1+m] + [1+m]\int_0^t c(s) + a(s)\big|\dot{x}(s)\big| + b(s)\big|x(s)\big|\,ds \\
&\leqslant M_0 + M_1 \cdot [1+m] + [1+m]\int_0^m c(s)\,ds + [1+m]\int_0^t k(s)\big(\big|x(s)\big| + \big|\dot{x}(s)\big|\big)\,ds,
\end{aligned}$$

where, for all $s \in [0,m]$, $k(s) := \max\{a(s), b(s)\}$.

By the Gronwall lemma (cf. [22]), we obtain that

$$\big|x(t)\big| + \big|\dot{x}(t)\big| \leqslant K \cdot e^{[1+m]\int_0^m k(s)\,ds}, \tag{4}$$

where

$$K := M_0 + [1+m]\left\{M_1 + \int_0^m c(s)\,ds\right\}.$$

Therefore, the set of solutions of (3) and their derivatives is bounded in $C^1([0,m],\mathbb{R}^n)$.

Let us now show that all solutions of (3) and their first derivatives are also equi-continuous. Let x be a solution of (3) and $t_2, t_3 \in [0,m]$ be arbitrary. Then, we have

$$
\begin{aligned}
&\left|x(t_3)-x(t_2)\right| \\
&\quad\leqslant \left|\dot{x}(t_0)\right|\cdot|t_3-t_2|+\left|\int_{t_0}^{t_3}(t_3-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds-\int_{t_0}^{t_2}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\quad= \left|\dot{x}(t_0)\right|\cdot|t_3-t_2|+\left|\int_{t_0}^{t_3}(t_3-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds-\int_{t_0}^{t_3}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right. \\
&\qquad\left.+\int_{t_0}^{t_3}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds-\int_{t_0}^{t_2}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\quad\leqslant \left|\dot{x}(t_0)\right|\cdot|t_3-t_2|+\left|\int_{t_0}^{t_3}(t_3-t_2)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right|+\left|\int_{t_3}^{t_2}(t_2-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\quad\leqslant \left|\dot{x}(t_0)\right|\cdot|t_3-t_2|+\left|\int_{t_0}^{t_3}|t_3-t_2|\cdot\left|P\big(s,x(s),\dot{x}(s)\big)\right|ds\right|+\left|\int_{t_3}^{t_2}|t_2-s|\cdot\left|P\big(s,x(s),\dot{x}(s)\big)\right|ds\right| \\
&\quad\leqslant M_1\cdot|t_3-t_2|+\left|\int_{t_0}^{t_3}|t_3-t_2|\cdot\big(c(s)+k(s)\cdot K\cdot e^{[1+m]\int_0^m k(u)\,du}\big)\,ds\right| \\
&\qquad+\left|\int_{t_3}^{t_2}|t_2-s|\cdot\big(c(s)+k(s)\cdot K\cdot e^{[1+m]\int_0^m k(u)\,du}\big)\,ds\right|. \qquad (5)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\left|\dot{x}(t_3)-\dot{x}(t_2)\right| &\leqslant \left|\int_{t_0}^{t_3}P\big(s,x(s),\dot{x}(s)\big)\,ds-\int_{t_0}^{t_2}P\big(s,x(s),\dot{x}(s)\big)\,ds\right| \\
&\leqslant \left|\int_{t_3}^{t_2}\left|P\big(s,x(s),\dot{x}(s)\big)\right|ds\right| \\
&\leqslant \left|\int_{t_3}^{t_2}\big(c(s)+k(s)\cdot K\cdot e^{[1+m]\int_0^m k(u)\,du}\big)\,ds\right|. \qquad (6)
\end{aligned}
$$

Taking into account estimates (5) and (6), x and $\dot{x}$ are equi-continuous, because $c(\cdot),k(\cdot)\in L^1([0,m],\mathbb{R})$. Thus, the set of solutions of (3) and their derivatives is relatively compact.

We will still show that the set of solutions of (3) and their derivatives is closed. Let $\{x_k\}$ be a sequence of solutions of (3) such that $(x_k,\dot{x}_k)\to(x,\dot{x})$. For all $k\in\mathbb{N}$ and a.a. $t\in[0,m]$, we have

$$\begin{aligned}\left|\dot{x}_k(t)\right| &\leqslant \left|\dot{x}_k(t_0)\right| + \left|\int_{t_0}^{t} \left|P\big(s, x_k(s), \dot{x}_k(s)\big)\right| ds\right| \\ &\leqslant M_1 + \left|\int_{t_0}^{t} \big(c(s) + k(s) \cdot K \cdot e^{[1+m]\int_0^m k(u)\,du}\big)\, ds\right|.\end{aligned}$$

Since $c(\cdot), k(\cdot) \in L^1([0,m],\mathbb{R})$, there exists a constant L such that, for a.a. $t \in [0,m]$,

$$\left|\int_{t_0}^{t} \big(c(s) + k(s) \cdot K \cdot e^{[1+m]\int_0^m k(u)\,du}\big)\, ds\right| \leqslant L.$$

Therefore, for all $k \in \mathbb{N}$ and for a.a. $t \in [0,m]$,

$$\left|\dot{x}_k(t)\right| \leqslant M_1 + L. \tag{7}$$

Moreover, since, for all $k \in \mathbb{N}$ and a.a. $t \in [0,m]$, $|\ddot{x}_k(t)| \leqslant c(t) + k(t) \cdot K \cdot e^{[1+m]\int_0^m k(u)\,du}$, the sequence $\{y_k := \dot{x}_k\}$ satisfies all assumptions of Lemma 2.1.

Thus, applying Lemma 2.1 to the sequence $\{\dot{x}_k\}$, we get that there exists a subsequence of $\{\dot{x}_k\}$, for the sake of simplicity denoted in the same way as the sequence, which converges uniformly to $\dot{x}$ on $[0,m]$ and such that $\{\ddot{x}_k\}$ converges weakly to $\ddot{x}$ in $L^1([0,m],\mathbb{R}^n)$.

If we set $z_k := (x_k, y_k)$, then $\dot{z}_k = (\dot{x}_k, \dot{y}_k) = (\dot{x}_k, \ddot{x}_k) \to (\dot{x}, \ddot{x})$ weakly in $L^1([0,m],\mathbb{R}^n)$. Let us now consider the system

$$\dot{z}_k(t) \in H\big(t, z_k(t)\big), \quad \text{for a.a. } t \in [0,m], \tag{8}$$

where $\dot{z}_k(t) = (\dot{x}_k(t), \dot{y}_k(t))$ and $H(t, z_k(t)) = (y_k(t), P(t, x_k(t), y_k(t)))$.

Applying Lemma 2.2, for $f_i := \dot{z}_k$, $f := (\dot{x}, \ddot{x})$, $x_i := z_k$, it follows that

$$\big(\dot{x}(t), \ddot{x}(t)\big) \in H\big(t, x(t), \dot{x}(t)\big),$$

for a.a. $t \in [0,m]$, i.e.

$$\ddot{x}(t) \in P\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0,m].$$

Moreover, since the set S'_m is closed, $(x_k, \dot{x}_k) \in S'_m$, for all $k \in \mathbb{N}$, and $(x_k, \dot{x}_k) \to (x, \dot{x})$, it also holds that $(x, \dot{x}) \in S'_m$. After all, the set of solutions of (3) and their derivatives is convex and compact, as claimed. □

Remark 3.1. If still $k \cdot B(t) \in C(t)$, for a.a. $t \in [0,m]$, and $k = (k_1, k_2, \ldots, k_n) \in S_m$, then constant k is obviously a solution of (2), and consequently the set of solutions of (2) is also nonempty. Nontrivial examples of solvability of (3), where S'_m corresponds to Kneser-type boundary conditions, are, for instance, in the scalar case ($n = 1$) the conditions $A(t) \equiv 1$, $C(t) - B(t)x \geqslant 0$, for $t \in [0,m]$, $x \in [0,\infty)$ (cf. [20]) or $C(t) \equiv 0$ and $B(t) \not\equiv 0$, $B(t) \leqslant 0$, for $t \in [0,m]$ (cf. Hartman–Wintner type results, e.g., in [27]).

Furthermore, let us study the structure of a solution set, on a compact interval, to a semi-linear problem.

Hence, let $m \in \mathbb{N}$ and let us consider the b.v.p.

$$\left.\begin{array}{r}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C\big(t,x(t),\dot{x}(t)\big),\quad \text{for a.a. } t\in[0,m],\\ l(x,\dot{x})=0,\end{array}\right\}\tag{P_m}$$

where

(i) $A, B \in L^1([0,m],\mathbb{R}^{n\times n})$ are such that $|A(t)| \leqslant a(t)$ and $|B(t)| \leqslant b(t)$, for all $t\in[0,m]$ and suitable integrable functions $a,b:[0,m]\to[0,\infty)$,
(ii) $l: C^1([0,m],\mathbb{R}^n)\times C([0,m],\mathbb{R}^n)\to\mathbb{R}^{2n}$ is a linear bounded operator,
(iii) the associated homogeneous problem

$$\left.\begin{array}{r}\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)=0,\quad \text{for a.a. } t\in[0,m],\\ l(x,\dot{x})=0\end{array}\right\}\tag{H_m}$$

has only the trivial solution,
(iv) $C:[0,m]\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping,
(v) there exists an integrable function $\alpha:[0,m]\to[0,\infty)$, with $\int_0^m \alpha(t)\,dt$ sufficiently small, such that

$$d_H\big(C(t,x_1,y_1),C(t,x_2,y_2)\big)\leqslant \alpha(t)\cdot\big(|x_1-x_2|+|y_1-y_2|\big),$$

for a.a. $t\in[0,m]$ and all $x_1,x_2,y_1,y_2\in\mathbb{R}^n$,
(vi) there exist a point $(x_0,y_0)\in\mathbb{R}^{2n}$ and a constant $C_0\geqslant 0$ such that

$$\big|C(t,x_0,y_0)\big|\leqslant C_0\cdot\alpha(t)$$

holds, for a.a. $t\in[0,m]$ ($\overset{\text{(v)}}{\Longrightarrow}$ $|C(t,x,y)|:=\sup\{|z| \mid z\in C(t,x,y)\}\leqslant \alpha(t)(C_0+|x_0|+|y_0|+|x|+|y|)$ holds, for a.a. $t\in[0,m]$ and all $x,y\in\mathbb{R}^n$).

Lemma 3.2. *Under the above assumptions* (i)–(vi)*, the set of solutions of the b.v.p.* (P_m) *is a nonempty, compact AR-space.*

Proof. Problem (P_m) is equivalent to the first-order problem

$$\left.\begin{array}{r}\dot{\xi}(t)+D(t)\xi(t)\in K\big(t,\xi(t)\big),\quad \text{for a.a. } t\in[0,m],\\ l(\xi)=0,\end{array}\right\}\tag{$\tilde{P}_m$}$$

where

$$\xi(t)_{2n\times 1}=\big(x(t),\dot{x}(t)\big)^T,$$
$$D(t)_{2n\times 2n}=\begin{pmatrix}0 & -I\\ B(t) & A(t)\end{pmatrix}$$

and

$$K(t,\xi)_{2n\times 1}=\big(0,C(t,x,\dot{x})\big)^T.$$

Similarly, the associated homogeneous problem (H_m) is equivalent to the first-order problem

$$\left.\begin{aligned} \dot{\xi}(t) + D(t)\xi(t) = 0, \quad & \text{for a.a. } t \in [0, m], \\ l(\xi) = 0. & \end{aligned}\right\} \tag{$\tilde{H}_m$}$$

The Fredholm alternative implies (see, e.g., [22]) that there exists the Green function $\tilde{G}$ for the homogeneous problem $(\tilde{H}_m)$ such that each solution $\xi(\cdot)$ of $(\tilde{P}_m)$ can be expressed by the formula $\xi(t) = \int_0^m \tilde{G}(t,s)k(s)\,ds$, where $k(\cdot)$ is a suitable measurable selection of $K(\cdot, \xi(\cdot))$ (cf. Proposition 2.2). If we denote by $\tilde{G}$ the block matrix

$$\tilde{G}_{2n\times 2n} = \begin{pmatrix} \tilde{G}^{11}_{n\times n} & \tilde{G}^{12}_{n\times n} \\ \tilde{G}^{21}_{n\times n} & \tilde{G}^{22}_{n\times n} \end{pmatrix}, \tag{9}$$

then each solution $x(\cdot)$ of (P_m) and its derivative $\dot{x}(\cdot)$ can be expressed as

$$x(t) = \int_0^m \tilde{G}^{12}(t,s)c(s)\,ds$$

and

$$\dot{x}(t) = \int_0^m \tilde{G}^{22}(t,s)c(s)\,ds,$$

where $c(\cdot)$ is a suitable measurable selection of $C(\cdot, x(\cdot), \dot{x}(\cdot))$. Moreover, in view of (v) and (vi),

$$\bigl|x(t)\bigr| + \bigl|\dot{x}(t)\bigr| \leqslant \int_0^m \bigl\{\bigl|\tilde{G}^{12}(t,s)\bigr| + \bigl|\tilde{G}^{22}(t,s)\bigr|\bigr\}\alpha(s)\bigl[C_0 + |x_0| + |y_0| + \bigl|x(s)\bigr| + \bigl|\dot{x}(s)\bigr|\bigr]\,ds,$$

for a.a. $t \in [0, m]$. If we denote by $\overline{G} := \sup_{(t,s)\in[0,m]\times[0,m]}\{|\tilde{G}^{12}(t,s)| + |\tilde{G}^{22}(t,s)|\}$, then

$$\max_{t\in[0,m]}\bigl\{\bigl|x(t)\bigr| + \bigl|\dot{x}(t)\bigr|\bigr\} \leqslant \overline{G}\int_0^m \alpha(s)\Bigl[C_0 + |x_0| + |y_0| + \max_{t\in[0,m]}\bigl\{\bigl|x(t)\bigr| + \bigl|\dot{x}(t)\bigr|\bigr\}\Bigr]\,ds.$$

Therefore,

$$\max_{t\in[0,m]}\bigl\{\bigl|x(t)\bigr| + \bigl|\dot{x}(t)\bigr|\bigr\} \leqslant \frac{\overline{G}\cdot(C_0 + |x_0| + |y_0|)\cdot\int_0^m \alpha(s)\,ds}{1 - \overline{G}\int_0^m \alpha(s)\,ds} := M,$$

provided

$$\int_0^m \alpha(s)\,ds < \frac{1}{\overline{G}}. \tag{10}$$

Therefore, if $\int_0^m \alpha(s)\,ds$ is small enough, namely if the inequality (10) holds, then the set of solutions of (P_m) is equal to the set of solutions of the problem

$$\left.\begin{aligned} \ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C^*\big(t,x(t),\dot{x}(t)\big), \quad &\text{for a.a. } t\in[0,m], \\ l(x,\dot{x})=0, \quad & \end{aligned}\right\} \tag{R_m}$$

where C^* satisfies conditions (iv)–(v) in Lemma 3.2 with C replaced by C^*, but this time

$$C^*(t,x,y):=\begin{cases} C(t,x,y), & \text{for } |x|\leqslant M \text{ and } |y|\leqslant M,\\ C(t,M_0,M_1), & \text{otherwise,}\end{cases}$$

where M_0, M_1 are suitable vectors such that $|M_0|=|M_1|=M$. It follows immediately from its definition that C^* satisfies

$$\begin{aligned} \big|C^*(t,x,y)\big| &:= \sup\big\{|z| \;\big|\; z\in C^*(t,x,y)\big\}=\sup\big\{|z| \;\big|\; z\in C(t,x,y),\ \text{where } |x|\leqslant M,\ |y|\leqslant M\big\}\\ &\leqslant \alpha(t)\big(C_0^*+\big|x_0^*\big|+\big|y_0^*\big|+2M\big):=\beta(t), \end{aligned} \tag{11}$$

where $(x_0^*, y_0^*)\in\mathbb{R}^{2n}$ is such that $|C^*(t,x_0^*,y_0^*)|\leqslant C_0^*\alpha(t)$, for a.a. $t\in[0,m]$.

Let us denote by $G(\cdot,\cdot\cdot):=\tilde{G}^{12}(\cdot,\cdot\cdot)$ the Green function associated to the second-order homogeneous problem (H_m) and define the Nemytskiĭ operator

$$N: C^1\big([0,m],\mathbb{R}^n\big)\multimap C^1\big([0,m],\mathbb{R}^n\big)$$

by the formula

$$Nx:=\Bigg\{h\in C^1\big([0,m],\mathbb{R}^n\big) \;\Bigg|\; h(\cdot)=\int_0^m G(\cdot,s)f(s)\,ds,\ \text{where } f\in L^1\big([0,m],\mathbb{R}^n\big),$$
$$f(t)\in C^*\big(t,x(t),\dot{x}(t)\big),\ \text{for a.a. } t\in[0,m]\Bigg\}.$$

Let us note that $Nx\neq\emptyset$, for all $x\in C^1([0,m],\mathbb{R}^n)$, because, for all $x\in C^1([0,m],\mathbb{R}^n)$, $C^*(t,x(t),\dot{x}(t))$ possesses a measurable selection (again, according to Proposition 2.2).

It is evident that the set of solutions of problem (R_m) is equal to the set of fixed-points of the operator N. In order to show that $\mathrm{Fix}(N)$ is, by means of Proposition 2.3, a nonempty, compact *AR*-space, we will proceed in several steps.

(1) At first, let us show that the operator N has convex values. If $h_1, h_2\in Nx$, then there exist integrable selections $f_1(\cdot), f_2(\cdot)$ of $C^*(\cdot,x(\cdot),\dot{x}(\cdot))$ such that, for a.a. $t\in[0,m]$,

$$h_1(t)=\int_0^m G(t,s)f_1(s)\,ds$$

and

$$h_2(t)=\int_0^m G(t,s)f_2(s)\,ds.$$

Let $\lambda \in [0,1]$ be arbitrary. Then, for a.a. $t \in [0,m]$,

$$\lambda h_1(t) + (1-\lambda)h_2(t) = \int_0^m G(t,s)\big[\lambda f_1(s) + (1-\lambda) f_2(s)\big]\,ds.$$

Since mapping C^* has convex values, $\lambda f_1(s) + (1-\lambda) f_2(s) \in C^*(s, x(s), \dot{x}(s))$, for a.a. $s \in [0,m]$. Therefore, $\lambda h_1 + (1-\lambda)h_2 \in Nx$, i.e. the operator N has convex values, as claimed.

(2) Secondly, let us show that the operator N has compact values. Let $x \in C^1([0,m], \mathbb{R}^n)$ be arbitrary and let v be an arbitrary integrable function such that $v(t) \in C^*(t, x(t), \dot{x}(t))$, for a.a. $t \in [0,m]$.

Let us consider the element h of Nx defined, for a.a. $t \in [0,m]$, by

$$h(t) := \int_0^m G(t,s)v(s)\,ds.$$

If $t, \tau \in [0,m]$ are arbitrary, then

$$\begin{aligned}\big|h(t) - h(\tau)\big| &= \left|\int_0^m G(t,s)v(s)\,ds - \int_0^m G(\tau,s)v(s)\,ds\right| \\ &\leqslant \int_0^m \big|G(t,s) - G(\tau,s)\big| \cdot \big|v(s)\big|\,ds \leqslant \int_0^m \big|G(t,s) - G(\tau,s)\big| \cdot \beta(s)\,ds. \qquad (12)\end{aligned}$$

Since $\beta(\cdot)$ is, by the definition, an integrable function, estimate (12) implies the equi-continuity of h. Moreover, it immediately follows from condition (11) and properties of the Green function that h is also bounded. Therefore, the well-known Arzelà–Ascoli lemma implies that the set Nx is relatively compact.

The relative compactness of values follows also alternatively from the contractivity of N which will be proved in the next step (3). It is namely well known that contractivity implies condensity.

The closedness of values follows from the fact that, according to [23], N can be expressed as the closed graph composition of operators $\phi \circ S_{C^*}$, where $S_{C^*} : C^1([0,m], \mathbb{R}^n) \multimap L^1([0,m], \mathbb{R}^n)$ and $\phi : L^1([0,m], \mathbb{R}^n) \to C^1([0,m], \mathbb{R}^n)$ are defined by

$$S_{C^*}(x) := \big\{ f \in L^1\big([0,m], \mathbb{R}^n\big) \;\big|\; f(t) \in C^*\big(t, x(t), \dot{x}(t)\big),\ \text{for a.a. } t \in [0,m] \big\}$$

and

$$\phi(f) := \left\{ h \in C^1\big([0,m], \mathbb{R}^n\big) \;\middle|\; h(t) = \int_0^m G(t,s) f(s)\,ds,\ \text{for a.a. } t \in [0,m] \right\}.$$

(3) In order to show that the operator N is a contraction, let us consider the Banach space $C^1([0,m], \mathbb{R}^n)$ endowed with the norm

$$|x|_{C^1} := \sup_{t \in [0,m]} \big\{ \big|x(t)\big| + \big|\dot{x}(t)\big| \big\},$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$. If $x, y \in C^1([0,m],\mathbb{R}^n)$ are arbitrary, then there exist $h_x \in Nx$, $h_y \in Ny$ and integrable selections $f_x(\cdot)$ of $C^*(\cdot, x(\cdot), \dot{x}(\cdot))$ and $f_y(\cdot)$ of $C^*(\cdot, y(\cdot), \dot{y}(\cdot))$ (cf. Proposition 2.2) such that

$$\begin{aligned}
d_H(Nx, Ny) &= |h_x - h_y|_{C^1} \\
&= \left| \int_0^m G(t,s) f_x(s)\,ds - \int_0^m G(t,s) f_y(s)\,ds \right|_{C^1} \\
&= \sup_{t\in[0,m]} \left\{ \left| \int_0^m G(t,s)\big[f_x(s) - f_y(s)\big]\,ds \right| + \left| \int_0^m \frac{\partial}{\partial t} G(t,s)\big[f_x(s) - f_y(s)\big]\,ds \right| \right\} \\
&\leqslant \sup_{t\in[0,m]} \int_0^m \left\{ \big|G(t,s)\big| + \left|\frac{\partial}{\partial t} G(t,s)\right| \right\} \cdot \big|f_x(s) - f_y(s)\big|\,ds \\
&\leqslant \sup_{t\in[0,m]} \big\{ \big|x(t) - y(t)\big| + \big|\dot{x}(t) - \dot{y}(t)\big| \big\} \\
&\quad \cdot \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ \big|G(t,s)\big| + \left|\frac{\partial}{\partial t} G(t,s)\right| \right\} \cdot \int_0^m \alpha(t)\,dt \\
&= \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ \big|G(t,s)\big| + \left|\frac{\partial}{\partial t} G(t,s)\right| \right\} \cdot \int_0^m \alpha(t)\,dt \cdot |x - y|_{C^1}.
\end{aligned}$$

If the integral $\int_0^m \alpha(t)\,dt$ is small enough, namely if

$$\mathfrak{L} := \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ \big|G(t,s)\big| + \left|\frac{\partial}{\partial t} G(t,s)\right| \right\} \cdot \int_0^m \alpha(t)\,dt < 1, \tag{13}$$

then the operator N is a desired contraction with a Lipschitz constant $\mathfrak{L} \in [0,1)$.

Finally, since N is a contraction with compact and convex values, the set $\mathrm{Fix}(N)$ is, according to Proposition 2.3, a nonempty, compact AR-space which completes the proof. □

Remark 3.2. The conclusion of Lemma 3.2 can be deduced from the main result for first-order systems in [8]. Our proof is, however, much more transparent and especially allows us to express explicitly the smallness of the integral $\int_0^m \alpha(t)\,dt$ in conditions (v) and (vi). It is given by the identical inequalities (10) and (13), namely

$$\int_0^m \alpha(t)\,dt < \frac{1}{\sup_{(t,s)\in[0,m]\times[0,m]}\{|G(t,s)| + |\frac{\partial}{\partial t} G(t,s)|\}}. \tag{14}$$

Remark 3.3. If the mapping $C(t,\cdot,\cdot)$ is Lipschitzian with a sufficiently small constant L, i.e. if condition (v) takes the form

(v') there exists a sufficiently small constant $L \geqslant 0$, such that

$$d_H\big(C(t,x_1,y_1),C(t,x_2,y_2)\big) \leqslant L\cdot\big(|x_1-x_2|+|y_1-y_2|\big),$$

for a.a. $t\in[0,m]$ and all $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, then the same conclusion holds, provided

$$L < \frac{1}{\sup_{t\in[0,m]}\int_0^m |G(t,s)|+|\frac{\partial}{\partial t}G(t,s)|\,ds}. \tag{15}$$

Example 3.1. Let us consider the Dirichlet problem

$$\left.\begin{array}{c}\ddot{x}(t)\in C\big(t,x(t),\dot{x}(t)\big),\quad \text{for a.a. } t\in[0,1],\\ x(0)=0,\quad x(1)=0,\end{array}\right\} \tag{16}$$

where $C:[0,m]\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that

$$d_H\big(C(t,x_1,y_1),C(t,x_2,y_2)\big) \leqslant \alpha(t)\cdot\big(|x_1-x_2|+|y_1-y_2|\big),$$

for a.a. $t\in[0,1]$, and all $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, where $\alpha\in L^1([0,1],[0,\infty))$ is such that (for better conditions, see [14,24] and cf. also [4, Theorem III.3.18])

$$\int_0^1 \alpha(t)\,dt < \frac{1}{2}. \tag{17}$$

Moreover, let there exist $C_0>0$ such that

$$\big|C(t,0,0)\big| \leqslant C_0\cdot\alpha(t),\quad \text{for a.a. } t\in[0,1]. \tag{18}$$

We will show that, under the above assumptions, the set of solutions of (16) is a nonempty, compact *AR*-space. The homogeneous problem associated to (16), i.e.

$$\left.\begin{array}{c}\ddot{x}(t)=0,\quad \text{for a.a. } t\in[0,1],\\ x(0)=0,\quad x(1)=0,\end{array}\right\}$$

has only the trivial solution and the related Green function G and its derivative $\frac{\partial G}{\partial t}$ take the forms

$$G(t,s):=\begin{cases}(t-1)s, & 0\leqslant t\leqslant s\leqslant 1,\\ (s-1)t, & 0\leqslant s\leqslant t\leqslant 1,\end{cases}$$

and

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}s, & 0\leqslant t\leqslant s\leqslant 1,\\ s-1, & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

Since

$$\sup_{(t,s)\in[0,1]\times[0,1]}\left\{\big|G(t,s)\big|+\left|\frac{\partial}{\partial t}G(t,s)\right|\right\}\leqslant 2,$$

condition (17) ensures that the problem (16) is, according to Lemma 3.2 (cf. condition (14)), solvable with a compact *AR*-space of solutions.

4. Topological structure on non-compact intervals

Because of counter-examples (see [1], [4, Example II.2.12], [16]), there is no chance to make a straightforward extension of Lemma 3.2 to b.v.p.s on non-compact intervals. On the other hand, the information concerning the solution sets on compact intervals can be sometimes useful for obtaining the topological structure of the set of solutions to asymptotic problems.

One of the efficient methods which can be used for studying b.v.p.s on non-compact intervals is an *inverse limit method*. Let us recall that by the *inverse system*, we mean a family $S=\{X_\alpha,\pi_\alpha^\beta,\Sigma\}$, where Σ is a set directed by the relation $\leqslant$, X_α is, for all $\alpha\in\Sigma$, a metric space and $\pi_\alpha^\beta : X_\beta\to X_\alpha$ is a continuous function, for all $\alpha,\beta\in\Sigma$ such that $\alpha\leqslant\beta$. Moreover, $\pi_\alpha^\alpha=\mathrm{id}_{X_\alpha}$ and $\pi_\alpha^\beta\pi_\beta^\gamma=\pi_\alpha^\gamma$, for all $\alpha\leqslant\beta\leqslant\gamma$. The *limit of inverse system* S is denoted by $\lim_\leftarrow S$ and it is defined by

$$\lim_{\leftarrow} S := \left\{(x_\alpha)\in\prod_{\alpha\in\Sigma}X_\alpha \;\middle|\; \pi_\alpha^\beta(x_\beta)=x_\alpha,\ \text{for all } \alpha\leqslant\beta\right\}.$$

If we denote by $\pi_\alpha : \lim_\leftarrow S\to X_\alpha$ the restriction of the projection $p_\alpha : \prod_{\alpha\in\Sigma}X_\alpha\to X_\alpha$ onto α-th axis, then it holds $\pi_\alpha=\pi_\alpha^\beta\pi_\beta$, for all $\alpha\leqslant\beta$.

Let us now consider two inverse systems $S=\{X_\alpha,\pi_\alpha^\beta,\Sigma\}$ and $S'=\{Y_{\alpha'},\pi_{\alpha'}^{\beta'},\Sigma'\}$. By a *multivalued mapping of the system* S *into the system* S', we mean a family $\{\sigma,\varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \Sigma'\to\Sigma$ and multivalued mappings $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')}\multimap Y_{\alpha'}$ such that, for all $\alpha'\leqslant\beta'$,

$$\pi_{\alpha'}^{\beta'}\varphi_{\sigma(\beta')}=\varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}^{\sigma(\beta')}.$$

Mapping $\{\sigma,\varphi_{\sigma(\alpha')}\}$ induces a *limit mapping* $\varphi : \lim_\leftarrow S\multimap\lim_\leftarrow S'$ satisfying, for all $\alpha'\in\Sigma'$,

$$\pi_{\alpha'}\varphi=\varphi_{\sigma(\alpha')}\pi_{\sigma(\alpha')}.$$

We will make use of the following result. For more details about the inverse limit method, see, e.g., [2–4,9,15].

Proposition 4.1. *(Cf. [3,18,25].) Let $S=\{X_m,\pi_m^p,\mathbb{N}\}$ and $S'=\{Y_m,\pi_m^p,\mathbb{N}\}$ be two inverse systems such that $X_m\subset Y_m$. If $\varphi : \lim_\leftarrow S\multimap\lim_\leftarrow S'$ is a limit map induced by a mapping $\{\mathrm{id},\varphi_m\}$, where $\varphi_m : X_m\multimap Y_m$, and if $\mathrm{Fix}(\varphi_m)$ are, for all $m\in\mathbb{N}$, R_δ-sets, then the fixed-point set $\mathrm{Fix}(\varphi)$ of φ is an R_δ-set, too.*

The following corollary is a direct consequence of Proposition 4.1.

Corollary 4.1. *Let us consider the sequence of b.v.p.s $\{(K_m)\}_{m=1}^\infty$, where*

$$\left.\begin{aligned} &\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C\big(t,x(t),\dot{x}(t)\big), \quad \text{for a.a. } t\in[t_0,t_0+m],\\ &k(x,\dot{x})|_{t\in[t_0,t_0+m]}=0, \end{aligned}\right\} \tag{K_m}$$

and let us assume that each problem (K_m), $m\in\mathbb{N}$, has an R_δ-set of solutions which corresponds to a fixed-point of the associated integral operator. Moreover, let the boundary condition be such that, for all $m\in\mathbb{N}$, the following holds:

If $x : [t_0,t_0+m]\to\mathbb{R}^n$ is a solution of problem (K_m), then $x|_{[t_0,t_0+m-1]} : [t_0,t_0+m-1]\to\mathbb{R}^n$ is a solution of problem (K_{m-1}).

Then the set of solutions of the problem

$$\left.\begin{aligned} &\ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C\big(t,x(t),\dot{x}(t)\big), \quad \text{for a.a. } t\in[t_0,\infty),\\ &k(x,\dot{x})=0 \end{aligned}\right\} \tag{K_∞}$$

is an R_δ-set.

As an illustration, we can give the following simple example.

Example 4.1. Consider the problem

$$\left.\begin{array}{c}\ddot{x}(t) \in C\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0, \infty),\\ x(0) = 0, \quad x(1) = 0,\end{array}\right\} \tag{19}$$

where $C : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that

$$d_H\big(C(t, x_1, y_1), C(t, x_2, y_2)\big) \leqslant \alpha(t) \cdot \big(|x_1 - x_2| + |y_1 - y_2|\big),$$

for a.a. $t \in [0, 1]$, and all $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$, where $\alpha \in L^1([0, 1], [0, \infty))$ is such that

$$\int_0^1 \alpha(t)\, dt < \frac{1}{2}. \tag{20}$$

Moreover, let there exist $C_0 > 0$ such that

$$\big|C(t, 0, 0)\big| \leqslant C_0 \cdot \alpha(t), \quad \text{for a.a. } t \in [0, 1]. \tag{21}$$

We will show that, under the above assumptions, the set of solutions of (19) can be expressed as a special union of R_δ-sets.

In order to solve (19), we will consider separately the Dirichlet problem

$$\left.\begin{array}{c}\ddot{x}(t) \in C\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [0, 1],\\ x(0) = 0, \quad x(1) = 0\end{array}\right\} \tag{22}$$

and the Cauchy (initial value) problem

$$\left.\begin{array}{c}\ddot{x}(t) \in C\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in [1, \infty),\\ x(1) = 0, \quad \dot{x}(1) = x_1.\end{array}\right\} \tag{23}$$

According to Lemma 3.2, the b.v.p. (22) is solvable with an R_δ-set of solutions (cf. Example 3.1). In fact, the set of solutions of (22) is, according to Lemma 3.2, a nonempty, compact *AR*-space.

Let $x(\cdot)$ be a solution of the Dirichlet problem (22) and let us put $x_1 := \dot{x}(1)$. Now, let us consider, for this inter-face value of the derivative, the problem (23). The Cauchy problem, considered on an arbitrary compact interval $[1, m]$, $m \in \mathbb{N}$, has an R_δ-set of solutions (cf. [13]). Using the inverse limit method, we can conclude that, for the fixed $x_1 = \dot{x}(1)$, the Cauchy problem (23) has, according to Corollary 4.1, an R_δ-set of solutions on $[1, \infty)$ which, in particular, implies that the related solution set is nonempty. If we denote by $x_D : [0, 1] \to \mathbb{R}^n$ the solution of the Dirichlet problem (22) satisfying $\dot{x}_D(1) = x_1$ and by $x_2 : [1, \infty] \to \mathbb{R}^n$ the solution of the Cauchy problem (23), then

$$x(t) := \begin{cases} x_D(t), & \text{for all } t \in [0, 1],\\ x_2(t), & \text{for all } t \in [1, \infty),\end{cases}$$

is the solution of the original problem (19).

Although the solution set of each separate problem was proved to be an R_δ-set, the solution set of the whole problem can be more complex. Nevertheless, if, for instance, the Dirichlet problem is uniquely solvable, then the solution set of the whole problem is an R_δ-set, too.

Combining Corollary 4.1 with Lemma 3.1, we obtain immediately the following result.

Proposition 4.2. *Let us consider the problems for fully linearized systems on compact intervals* (2) *and* (3) *together with the asymptotic problems*

$$\left.\begin{array}{c} \ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C(t), \quad \text{for a.a. } t\in[0,\infty), \\ x\in S, \end{array}\right\} \tag{24}$$

$$\left.\begin{array}{c} \ddot{x}(t)+A(t)\dot{x}(t)+B(t)x(t)\in C(t), \quad \text{for a.a. } t\in[0,\infty), \\ (x,\dot{x})\in S', \end{array}\right\} \tag{25}$$

where

(i) *$A, B : [0,\infty)\to\mathbb{R}^{n\times n}$ are locally integrable matrix functions such that $|A(t)|\leqslant a(t)$, $|B(t)|\leqslant b(t)$, for a.a. $t\in[0,\infty)$ and a suitable nonnegative functions $a, b\in L^1_{loc}([0,\infty),\mathbb{R})$,*

(ii) *S and S_m are, for all $m\in\mathbb{N}$, closed, convex subsets of $AC^1_{loc}([0,\infty),\mathbb{R}^n)$ and $AC^1([0,m],\mathbb{R}^n)$ (S' and S'_m are, for all $m\in\mathbb{N}$, closed, convex subsets of $AC^1_{loc}([0,\infty),\mathbb{R}^n)\times AC_{loc}([0,\infty),\mathbb{R}^n)$ and $AC^1([0,m],\mathbb{R}^n)\times AC([0,m],\mathbb{R}^n)$),*

(iii) *$C : [0,\infty)\multimap\mathbb{R}^n$ is a locally integrable mapping with convex closed values such that $|C(t)|\leqslant c(t)$, for a.a. $t\in[0,m]$ and a suitable nonnegative function $c\in L^1_{loc}([0,\infty),\mathbb{R})$,*

(iv) *there exists $t_0\in[0,\infty)$ such that, for all $m\in\mathbb{N}$, we are able to find constants M_{m_0}, M_{m_1} such that $|x(t_0)|\leqslant M_{m_0}$ and $|\dot{x}(t_0)|\leqslant M_{m_1}$, for all solutions $x(\cdot)$ of problem* (2) (*all solutions $x(\cdot)$ of problem* (3)).

Moreover, let, for all $m\in\mathbb{N}$, the set of solutions of (2) (*the set of solutions of* (3) *and their derivatives*) *be nonempty and correspond to fixed-points of the associated integral operator. Furthermore, let the boundary condition be such that, for all $m\in\mathbb{N}$, the following holds*:

If $x : [0,m]\to\mathbb{R}^n$ belongs to S_m, then $x|_{[0,m-1]} : [0,m-1]\to\mathbb{R}^n$ belongs to S_{m-1}.

(*If $(x,\dot{x}) : [0,m]\times[0,m]\to\mathbb{R}^{2n}$ belongs to S'_m, then $(x|_{[0,m-1]},\dot{x}|_{[0,m-1]}) : [0,m-1]\times[0,m-1]\to\mathbb{R}^{2n}$ belongs to S'_{m-1}.*)

Then the set of solutions of the problem (24) (*the set of solutions of the problem* (25) *and their derivatives*) *is an R_δ-set.*

As an application of Proposition 4.2, let us consider the second-order asymptotic (Kneser-type) b.v.p. with a multivalued vector perturbation

$$\left.\begin{array}{c} \ddot{x}(t)+\dot{x}(t)+b(t)x(t)\in F(t), \quad \text{for a.a. } t\in[0,\infty), \\ x(0)=1, \\ x(t)\geqslant 0, \quad \dot{x}(t)\leqslant 0, \quad \text{for all } t\in[0,\infty), \end{array}\right\} \tag{P}$$

where

- $b : [0,\infty)\to\mathbb{R}$ is a locally integrable function such that $|b(t)|\leqslant a(t)$, for all $t\in[0,\infty)$, where $a\in L^1_{loc}([0,\infty),\mathbb{R})$,
- $F : [0,\infty)\multimap\mathbb{R}$ is a locally integrable mapping with convex closed values such that $|F(t)|\leqslant\alpha(t)$, for all $t\in[0,\infty)$, where $\alpha\in L^1_{loc}([0,\infty),\mathbb{R})$ and that

$$-b(t)\notin F(t), \quad \text{for all } t \text{ in a right neighbourhood of } 0.$$

Moreover, let $v(t)-b(t)x\geqslant 0$, for all $t\in[0,\infty)$, $x\in[0,1]$ and each measurable selection $v(\cdot)$ of $F(\cdot)$.

We will prove the following theorem.

Theorem 4.1. *Under the above assumptions, the set of solutions of* (P) *and their first derivatives is an* R_δ*-set.*

Proof. Together with the b.v.p. (P), let us consider the family of associated problems on compact intervals

$$\left.\begin{array}{c} \ddot{x}(t)+\dot{x}(t)+b(t)\cdot x(t)\in F(t), \quad \text{for a.a. } t\in[0,m], \\ x(0)=1, \\ x(t)\geqslant 0, \quad \dot{x}(t)\leqslant 0, \quad \text{for all } t\in[0,m], \end{array}\right\} \tag{P_m}$$

where $m\in\mathbb{N}$.

It was shown in [11] (see Lemma 2.1 in [11] and the remarks below) that under the above assumptions imposed on b, the following two norms in $AC^1([0,m],\mathbb{R})$, where $m>0$ is arbitrary, are equivalent:

$$\|x\|:=\sup_{t\in[0,m]}\left|x(t)\right|+\sup_{t\in[0,m]}\left|\dot{x}(t)\right|+\int_0^m\left|\ddot{x}(t)\right|dt;$$

$$\|x\|_*:=\sup_{t\in[0,m]}\left|x(t)\right|+\int_0^m\left|\ddot{x}(t)+b(t)\cdot x(t)+\dot{x}(t)\right|dt.$$

If x is a solution of the b.v.p. (P_m), for some $m\in\mathbb{N}$, then

$$\|x\|_*=1+\int_0^m\alpha(t)\,dt:=M_m,$$

because α is a locally integrable function.

Since $\sup_{t\in[0,m]}|\dot{x}(t)|\leqslant\|x\|$, and the equivalence of norms $\|x\|_*$ and $\|x\|$, there exists the sequence $\{k_m\}$ of positive numbers such that all solutions of the b.v.p. (P_m), for fixed $m\in\mathbb{N}$, satisfy $|\dot{x}(t)|\leqslant k_m\cdot M_m$.

Since the sets

$$\begin{aligned} S'_m:=\big\{&(x,\dot{x})\in AC^1\big([0,m],\mathbb{R}\big)\times AC\big([0,m],\mathbb{R}\big),\ x(0)=1, \\ &x(t)\geqslant 0,\ \dot{x}(t)\leqslant 0,\ \text{for all } t\in[0,m]\big\}, \\ S':=\big\{&(x,\dot{x})\in AC^1_{loc}\big([0,\infty),\mathbb{R}\big)\times AC_{loc}\big([0,\infty),\mathbb{R}\big),\ x(0)=1, \\ &x(t)\geqslant 0,\ \dot{x}(t)\leqslant 0,\ \text{for all } t\in[0,\infty)\big\} \end{aligned}$$

are closed and convex, the b.v.p.s (P), (P_m) satisfy assumptions (i)–(iv) of Proposition 4.2 (with $M_{m_0}=1, M_{m_1}=k_m\cdot M_m$).

The non-emptiness of the set of solutions of (P_m) follows from Corollary 2 in [20] and the fact that $F(\cdot)$ admits (according to Proposition 2.2) a single-valued measurable selection $v(\cdot)$ such that $v(t)-b(t)x\geqslant 0$, for all $t\in[0,\infty)$ and $x\in[0,\infty)$.

If we denote by $P(t,x(t),\dot{x}(t)):=F(t)-b(t)x(t)-\dot{x}(t)$, then $x(\cdot)$ is a solution of (P_m) if and only if, for a.a. $t\in[0,m]$,

$$x(t)\in x(u)-\left|x(u)\right|+1+\dot{x}(0)\cdot t+\int_0^t(t-s)\cdot P\big(s,x(s),\dot{x}(s)\big)\,ds, \tag{26}$$

$$\dot{x}(t) \in \dot{x}(u) + \left|\dot{x}(u)\right| + \dot{x}(0) + \int_0^t P\bigl(s, x(s), \dot{x}(s)\bigr)\, ds, \tag{27}$$

for each $u \in [0, m]$, provided

$$-b(t) \notin F(t), \tag{28}$$

on a subset of $[0, m]$ with a non-zero measure. Indeed. Since the constraint in (P_m) can be equivalently expressed as

$$\left.\begin{aligned} x(0) = 1, \\ x(u) - \bigl|x(u)\bigr| = 0, \quad \dot{x}(u) + \bigl|\dot{x}(u)\bigr| = 0, \quad \text{for all } u \in [0, m], \end{aligned}\right\} \tag{29}$$

every solution $x(\cdot)$ of (P_m) and its derivative $\dot{x}(\cdot)$ obviously satisfy (26) and (27). Reversely, derivating (26) and (27), we obtain

$$\dot{x}(t) \in \dot{x}(0) + \int_0^t P\bigl(s, x(s), \dot{x}(s)\bigr)\, ds,$$

$$\ddot{x}(t) \in P\bigl(t, x(t), \dot{x}(t)\bigr).$$

Moreover, $x(0) \in x(u) - |x(u)| + 1$, $\dot{x}(0) \in \dot{x}(u) + |\dot{x}(u)| + \dot{x}(0)$, for each $u \in [0, m]$, i.e. $\dot{x}(u) + |\dot{x}(u)| = 0$ and, in particular, for $u = 0$, $|x(0)| = 1$. Thus, for $x(0) = 1$, we also have $x(u) - |x(u)| = 0$, by which (29) (i.e. the constraint in (P_m)) is satisfied. On the other hand, if $x(0) = -1$, we arrive at $x(u) - |x(u)| = -2$, i.e. $x(u) = -1$, for all $u \in [0, m]$, and subsequently $-b(t) \in F(t)$, for a.a. $t \in [0, m]$, which is a contradiction with (28).

The set of solutions of (P_m) and their first derivatives is a fixed-point set of the map $\varphi_m : C^1([0, m], \mathbb{R}) \times C([0, m], \mathbb{R}) \multimap C^1([0, m], \mathbb{R}) \times C([0, m], \mathbb{R})$, where, for all $t \in [0, m]$,

$$\begin{aligned} \varphi_m(x, \dot{x})(t) := \Biggl\{ \Biggl(& \bigcup_{u \in [0,m]} x(u) - \bigl|x(u)\bigr| + 1 + \dot{x}(0) \cdot t + \int_0^t (t - s) \cdot f(s)\, ds, \\ & \bigcup_{u \in [0,m]} \dot{x}(u) + \bigl|\dot{x}(u)\bigr| + \dot{x}(0) + \int_0^t f(s)\, ds \Biggr) \Bigm| f \in L^1\bigl([0, m], \mathbb{R}^n\bigr) \text{ and} \\ & f(s) \in P\bigl(t, x(s), \dot{x}(s)\bigr), \text{ for a.a. } s \in [0, m] \Biggr\}. \end{aligned}$$

It can be easily seen that $\{\varphi_m\}_{m=1}^{\infty}$ is a map of the inverse system

$$\bigl\{C^1\bigl([0, m], \mathbb{R}\bigr) \times C\bigl([0, m], \mathbb{R}\bigr), \pi_m^p, \mathbb{N}\bigr\}$$

into itself, where, for all $p \geqslant m$, $x \in C^1([0, p], \mathbb{R}) \times C([0, p], \mathbb{R})$, $\pi_m^p(x, \dot{x}) = (x|_{[0,m]}, \dot{x}|_{[0,m]})$. Mappings $\{\varphi_m\}_{m=1}^{\infty}$ induce the limit mapping $\varphi : C^1([0, \infty), \mathbb{R}) \times C([0, \infty), \mathbb{R}) \multimap C^1([0, \infty), \mathbb{R}) \times C([0, \infty), \mathbb{R})$, where, for all $t \geqslant 0$,

$$\varphi(x,\dot{x})(t) := \left\{ \left(\bigcup_{u\in[0,\infty)} x(u) - \left|x(u)\right| + 1 + \dot{x}(0)\cdot t + \int_0^t (t-s)\cdot f(s)\,ds, \right. \right.$$
$$\left. \bigcup_{u\in[0,\infty)} \dot{x}(u) + \left|\dot{x}(u)\right| + \dot{x}(0) + \int_0^t f(s)\,ds \right) \Bigm| f \in L^1\big([0,m],\mathbb{R}^n\big) \text{ and}$$
$$\left. f(s) \in P\big(t, x(s), \dot{x}(s)\big),\ \text{for a.a. } s \in [0,\infty) \right\}.$$

The fixed-point set of the mapping φ is the set of solutions and their derivatives of the problem (P). Applying Proposition 4.2, the set of solutions and their first derivatives of the original problem (P) is therefore an R_δ-set, as claimed. □

Remark 4.1. One can readily check that if $(1,\ldots,1)\cdot B(t) \in C(t)$, for a.a. $t \in [0,\infty)$, then constant vector $(1,\ldots,1)$ is a stationary solution of the Kneser-type asymptotic problem

$$\left.\begin{array}{c} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)\cdot x(t) \in C(t), \quad \text{for a.a. } t \in [0,\infty), \\ x_i(0) = 1, \quad \text{for all } i \in \{1,2,\ldots,n\}, \\ x_i(t) \geqslant 0, \quad \dot{x}_i(t) \leqslant 0, \quad \text{for all } i \in \{1,2,\ldots,n\} \text{ and } t \in [0,\infty), \end{array}\right\} \qquad (P^1)$$

where

- $A : [0,\infty) \to \mathbb{R}^{n\times n}$ is a locally integrable matrix function such that $|A(t)| \leqslant a(t)$, for all $t \in [0,\infty)$, where $a \in L^1_{loc}([0,\infty),\mathbb{R})$,
- $B : [0,\infty) \to \mathbb{R}^{n\times n}$ is a locally integrable matrix function such that $|B(t)| \leqslant b(t)$, for all $t \in [0,\infty)$, where $b \in L^1_{loc}([0,\infty),\mathbb{R})$,
- $C : [0,\infty) \multimap \mathbb{R}^n$ is a locally integrable mapping with convex closed values such that $|C(t)| \leqslant c(t)$, for all $t \in [0,\infty)$, where $c \in L^1_{loc}([0,\infty),\mathbb{R})$.

If still

$$(-1,\ldots,-1)\cdot B(t) \notin C(t), \quad \text{for a.a. } t \text{ in a right neighborhood of } 0,$$

then it can be proved quite analogously as in Theorem 4.1 that the set of solutions of (P^1) and their first derivatives is an R_δ-set.

Remark 4.2. Similarly, if $a, b \in L^1_{loc}([0,\infty),\mathbb{R})$ are locally integrable functions such that $b(t) \leqslant 0$, for a.a. $t \in [0,\infty)$, and $b(t) \neq 0$, for a.a. t in a right neighborhood of 0, then (cf. Remark 3.1) it can be proved quite analogously as in Theorem 4.1 that the set of solutions of the Kneser-type asymptotic problem

$$\left.\begin{array}{c} \ddot{x}(t) + a(t)\dot{x}(t) + b(t)\cdot x(t) = 0, \quad \text{for a.a. } t \in [0,\infty), \\ x(0) = 1, \\ x(t) \geqslant 0, \quad \dot{x}(t) \leqslant 0, \quad \text{for all } t \in [0,\infty), \end{array}\right\}$$

and their first derivatives is an R_δ-set.

Remark 4.3. As a particular case of Corollary 4.1, we are theoretically able to obtain the result which can be proved combining Lemma 3.2 and Proposition 4.1. In such a case an integrable function α in condition (v) in Lemma 3.2 should however satisfy conditions (cf. (14))

$$\int_{t_0}^{t_0+m} \alpha(t)\,dt < \frac{1}{\sup_{(t,s)\in[t_0,t_0+m]\times[t_0,t_0+m]}\{|G_m(t,s)| + |\frac{\partial}{\partial t}G_m(t,s)|\}}, \tag{30}$$

for sufficiently large $m \in \mathbb{N}$, which is probably not very realistic.

5. Application to existence result

As we could see, the investigation of a topological structure of solution sets to asymptotic problems is sufficiently interesting itself. Nevertheless, the main advantage consists in its further application to existence results for nonlinear asymptotic problems.

This application will be demonstrated here by means of the following proposition developed by ourselves in [5, Theorem 3.1 and Corollary 4.1].

Proposition 5.1. *Let us consider the b.v.p.*

$$\left.\begin{array}{c} \ddot{x}(t) \in F\big(t, x(t), \dot{x}(t)\big), \quad \text{for a.a. } t \in J, \\ x \in S, \end{array}\right\} \tag{31}$$

where J is a given (possibly infinite) real interval, $F : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $AC^1_{loc}(J, \mathbb{R}^n)$.

Let $H : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory map such that

$$H(t, c, d, c, d) \subset F(t, c, d), \quad \text{for all } (t, c, d) \in J \times \mathbb{R}^n \times \mathbb{R}^n.$$

Assume that

(i) *there exists a retract Q of $C^1(J, \mathbb{R}^n)$ such that the associated problem*

$$\left.\begin{array}{c} \ddot{x}(t) \in H\big(t, x(t), \dot{x}(t), q(t), \dot{q}(t)\big), \quad \text{for a.a. } t \in J, \\ x \in S \cap Q, \end{array}\right\} \tag{32}$$

is solvable with an R_δ-set of solutions, for each $q \in Q$,

(ii) *there exists a nonnegative, locally integrable function $\alpha : J \to \mathbb{R}$ such that*

$$\big|H\big(t, x(t), \dot{x}(t), q(t), \dot{q}(t)\big)\big| \leqslant \alpha(t)\big(1 + \big|x(t)\big| + \big|\dot{x}(t)\big|\big), \quad \text{a.e. in } J,$$

for any $(q, x) \in \Gamma_T$, where T denotes the multivalued map which assigns to any $q \in Q$ the set of solutions of (32),

(iii) $\overline{T(Q)} \subset S$,

(iv) *there exist a point $t_0 \in J$ and constants $M_0 \geqslant 0$, $M_1 \geqslant 0$ such that $|x(t_0)| \leqslant M_0$ and $|\dot{x}(t_0)| \leqslant M_1$, for any $x \in T(Q)$.*

Then the b.v.p. (31) has a solution in $S \cap Q$.

Hence, let us consider the second-order nonlinear (Kneser-type) asymptotic b.v.p.

$$\left.\begin{array}{c}\ddot{x}(t)+\dot{x}(t)+B\big(t,x(t),\dot{x}(t)\big)\cdot x(t)\in F\big(t,x(t),\dot{x}(t)\big),\quad \text{for a.a. } t\in[0,\infty),\\ x_i(0)=1,\quad \text{for all } i\in\{1,2,\dots,n\},\\ x_i(t)\geqslant 0,\quad \dot{x}_i(t)\leqslant 0,\quad \text{for all } i\in\{1,2,\dots,n\} \text{ and } t\in[0,\infty),\end{array}\right\} \tag{33}$$

where

- $x(t)=(x_1(t),x_2(t),\dots,x_n(t))$,
- $B:[0,\infty)\times\mathbb{R}^{2n}\to\mathbb{R}^{n\times n}$ is a diagonal Carathéodory matrix function, i.e.

$$B\big(t,x(t),\dot{x}(t)\big)=\begin{pmatrix} b_1(t,x(t),\dot{x}(t)) & 0 & \dots & 0\\ 0 & b_2(t,x(t),\dot{x}(t)) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & b_n(t,x(t),\dot{x}(t))\end{pmatrix}$$

  with $|B(t,x,y)|\leqslant\beta(t)(1+|x|)$, for all $(x,y)\in\mathbb{R}^{2n}$ and $t\in[0,\infty)$, where $\beta\in L^1_{loc}([0,\infty),\mathbb{R})$,
- $F=(F_1,F_2,\dots,F_n):[0,\infty)\times\mathbb{R}^{2n}\multimap\mathbb{R}^n$ is an upper-Carathéodory mapping such that $|F(t,x,y)|\leqslant\alpha(t)(1+|x|)$, for all $(x,y)\in\mathbb{R}^{2n}$ and $t\in[0,\infty)$, where $\alpha\in L^1_{loc}([0,\infty),\mathbb{R})$, and that

$$-b_i(t,x,y)\notin F_i(t,x,y),\quad \text{for all } i\in\{1,\dots,n\},\ (x,y)\in[0,1]^n\times(-\infty,0]^n$$

  and for t in a right neighbourhood of 0.

For applying Proposition 5.1, let us define the set of candidate solutions as follows

$$Q:=\Big\{(x_1,\dots,x_n)\in C^1\big([0,\infty),\mathbb{R}^n\big)\ \Big|\ x_i(0)=1,\ x_i(t)\geqslant 0,\ \dot{x}_i(t)\leqslant 0,\\ \text{for all } i\in\{1,\dots,n\},\ t\in[0,\infty)\Big\}.$$

Let us still consider the associated problems

$$\left.\begin{array}{c}\ddot{x}(t)+\dot{x}(t)+B\big(t,q(t),\dot{q}(t)\big)\cdot x(t)\in F\big(t,q(t),\dot{q}(t)\big),\quad \text{for a.a. } t\in[0,\infty),\\ x_i(0)=1,\quad \text{for all } i\in\{1,2,\dots,n\},\\ x_i(t)\geqslant 0,\quad \dot{x}_i(t)\leqslant 0,\quad \text{for all } i\in\{1,2,\dots,n\} \text{ and } t\in[0,\infty).\end{array}\right\} \tag{P_q}$$

If $v_i(t)-b_i(t,q(t),\dot{q}(t))\cdot x_i\geqslant 0$, for all $i\in\{1,2,\dots,n\}$, $q\in Q$, $t\in[0,\infty)$, $x_i\in[0,1]$ and each measurable selection $v_i(t)$ of $F_i(t,q(t),\dot{q}(t))$, then we will check that the b.v.p. (33) has a solution.

More concretely, let us verify, that the b.v.p. (P_q) satisfies, for all $q\in Q$, all assumptions of Proposition 5.1.

- ad (i) Since (P_q) represents n separate problems on a diagonal, it can be proved exactly in the same way as in Theorem 4.1 that the b.v.p. (P_q) has, for each $q\in Q$, an R_δ-set of solutions.
- ad (ii) $|F(t,q(t),\dot{q}(t))-B(t,q(t),\dot{q}(t))\cdot x-y|\leqslant\alpha(t)(1+\sqrt{n})+\beta(t)\cdot(1+\sqrt{n})\cdot|x|+|y|$, for a.a. $t\in[0,\infty)$, all $(x,y)\in\mathbb{R}^{2n}$ and $q\in Q$.
- ad (iii) Since the set $S:=Q$ is closed and each solution of the b.v.p. (P_q) belongs to Q, it holds that $\overline{T(Q)}\subset S$, where the map T is the solution mapping that assigns to each $q\in Q$ the set of solutions of (P_q).

ad (iv) Let $t_0 = 0$. Then each solution $x(\cdot)$ of (P_q) satisfies, for an arbitrary $q \in Q$, $|x(0)| = \sqrt{n}$. Moreover, the fact that $x(\cdot)$ is a solution of (P_q) implies that $x(\cdot)$ is also a solution of the b.v.p.

$$\left.\begin{aligned} &\ddot{x}(t) + \dot{x}(t) + B\big(t, q(t), \dot{q}(t)\big) \cdot x(t) \in F\big(t, q(t), \dot{q}(t)\big), \quad \text{for a.a. } t \in [0,1],\\ &x_i(0) = 1, \quad \text{for all } i \in \{1, 2, \ldots, n\},\\ &x_i(t) \geqslant 0, \quad \dot{x}_i(t) \leqslant 0, \quad \text{for all } i \in \{1, 2, \ldots, n\} \text{ and } t \in [0,1]. \end{aligned}\right\} \qquad (P_{q,1})$$

Thus, it follows from the arguments in the proof of Theorem 4.1 that $|\dot{x}(0)| \leqslant k_1 \cdot M_1$, where k_1 is a suitable positive constant and $M_1 := \sqrt{n} + \int_0^1 \alpha(t)\,dt$.

Since all assumptions of Proposition 5.1 are satisfied, we are ready to formulate the last theorem.

Theorem 5.1. *Under the above assumptions, the b.v.p.* (33) *admits a solution* $x(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot))$ *such that* $0 \leqslant x_i(t) \leqslant 1$, *for all* $i \in \{1, 2, \ldots, n\}$ *and* $t \in [0, \infty)$.

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